MA4211
Functional Analysis

Tools

- **Young's ineq.**: Given $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ then $\forall a, b \in \mathbb{C}, |ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}$
- **Hölder's ineq.**: Given $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ then $\forall \mathbf{x}, \mathbf{y} \in \mathbb{C}^n, \sum_{i=1}^n |x_i y_i| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q$ (holds when $n = \infty$ too)
  Integral version:
  $\int_a^b |f(x)g(x)|\, dx \leq \left(\int_a^b |f(x)|^p\, dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\, dx\right)^{\frac{1}{q}}$
- **Minkowski's ineq.**: Given $p > 1$ then $\forall \mathbf{x}, \mathbf{y} \in \mathbb{C}^n, \|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p$
  (also works for ℓ^p and L^p spaces)
- **Zorn's lemma**: Suppose a partially ordered set P has the property that every chain in P has an upper bound in P. Then the set P contains at least one maximal element.

Vector Spaces

- TODO: vect. space axioms
- $x_1, \dots, x_n$ **(fin.) are lin. indep.**: $\alpha_1 x_1 + \dots + \alpha_n x_n = 0 \implies \alpha_1 = \dots = \alpha_n = 0$
- Σ **spans** V: $\forall v \in V$, v is (fin.) lin. combin. of vectors in V
- $M \subseteq X$ **is basis for** X: any fin. set of vectors in M are lin. indep., and M spans X
- **Equiv. basis**: Σ is a basis of V
  $\iff$ Σ is a maximal lin. indep. subset of V
  $\iff$ Σ is a minimal spanning subset of V
- **Existence of basis**: every vector space has a basis
  - Proof using Zorn's lemma: $\Sigma \coloneqq \{B \subseteq V \mid B \text{ is lin. indep.}\}$
- **Cardinality of basis**: all bases of V have same cardinality
- X **is finite-dimensional**: bases of X has finite cardinality

Metric Spaces

- **Def**: set X with distance d satisfying:
  - d is real-valued, finite, and non-negative
  - $d(x, y) = 0 \iff x = y$
  - $d(x, y) = d(y, x)$ (symmetry)
  - $d(x, y) \leq d(x, z) + d(z, y)$ ($\triangle$ ineq.)
- **Equiv. continuity**: $T : X \to Y$ is cts.
  $\iff \forall$ open set $S \subseteq Y$, $T^{-1}(S)$ is open subset of X
- Convergent $\implies$ Cauchy
  In complete metric spaces: Convergent $\iff$ Cauchy
- **Isometry**: distance-preserving transformation
- **Complete**: Every Cauchy sequence converges
- Every Cauchy sequence is bounded
- **Continuous** $\iff$ every open set has an open pre-image
  $\iff$ every closed set has a closed pre-image
- $f : X \to Y$ is a **homeomorphism**: f bijective; f & f^{-1} cts

Normed Spaces

- **Def**: set X with norm $\|\cdot\|$ satisfying:
  - $\|x\| \geq 0$
  - $\|x\| = 0 \iff x = 0$
  - $\|\alpha x\| = |\alpha|\|x\|$ (scaling)
  - $\|x + y\| \leq \|x\| + \|y\|$ ($\triangle$ ineq.)
  Normed space gives rise to a metric $d(x, y) \coloneqq \|x - y\|$
- **Normed space**: vector space with norm (normed space is cts. in both arguments for both VA and SM)
- **Banach space**: normed spaces whose metric is complete
- **Sequence spaces**: $\ell^p \coloneqq \left\{\mathbf{a} \in \mathbb{R}^{\mathbb{N}} \mid \sum_{i=0}^{\infty} |a_i|^p < \infty\right\}$
  $\ell^{\infty} \coloneqq \left\{\mathbf{a} \in \mathbb{R}^{\mathbb{N}} \mid \sup_{i=0}^{\infty} |a_i| < \infty\right\}$
  are Banach spaces
- **Thm**: subspace Y of Banach space X is complete $\iff$ Y is closed wrt X
- **Isometry**: distance-preserving transformation
- **Completion thm**: Given normed space X, there is a Banach space $\hat{X}$ and an isometry A from X onto a subspace W of $\hat{X}$ which is dense in $\hat{X}$. The space $\hat{X}$ is unique up to isometry.
  (Metric completion applies for general metric spaces too)
- $\|\cdot\|$ and $\|\cdot\|'$ are **equivalent**:
  $\exists a, b > 0$ such that $\forall v \in V, a\|v\| \leq \|v\|' \leq b\|v\|$
- **Finite-dim vector spaces**: On any finite-dim V (over $\mathbb{R}$ or $\mathbb{C}$):
  - Any two norms are equivalent
  - $B(1)$ is compact
  - V is complete (hence Banach)
  - Any subspace $W \subseteq V$ is complete (hence closed in V)
- **Function spaces**: $\|f\|_p \coloneqq \left(\int_a^b |f(x)|^p\, dx\right)^{\frac{1}{p}}$
  Facts:
  - $p = \infty$: $(C[0,1], \|\cdot\|_{\infty})$ Banach
  - $p < \infty$: $(C[0,1], \|\cdot\|_p)$ not complete
  - $(L^p[0,1], \|\cdot\|_p) \coloneqq (V/V_0, \|\cdot\|_p)$ is a (dense) completion of $(C[0,1], \|\cdot\|_p)$ where
  $V \coloneqq \left\{f : [0,1] \to \mathbb{R} \mid f \text{ measurable}, \int_0^1 |f(x)|^p\, dx < \infty\right\}$ and
  $V_0 \coloneqq \left\{f \in V \mid \int_0^1 |f(x)|^p\, dx = 0\right\}$ (in other words, $L^p[0,1]$ is like V, but where functions that agree almost everywhere are identified)

Linear Operators

- **Def**: $T : V \to W$ where V, W are vector spaces over F and $T(x + y) = Tx + Ty$ [VA] and $T(\alpha x) = \alpha Tx$ [SM]
- **E.g. Integration as lin. op.**: $T : L^1([a,b]) \to L^1([a,b])$ where $(Tx)(t) \coloneqq \int_a^t x(\tau) d\tau$ is a lin. op.
- **Subspaces**: lin. op. $T : X \to Y$:
  $T(X)$ is a subspace of Y; $\mathrm{Ker}(T)$ is a subspace of X
- **Inverse**: lin. op. $T : X \to Y$:
  - T is injective $\iff$ $(Tx = 0 \implies x = 0)$
  - if T^{-1} exists, then it is a lin. op. $T^{-1} : Y \to X$
- lin. op. $T : X \to Y$ is **bounded**:
  $\exists c > 0$ such that $\forall x \in X, \|Tx\| \leq c\|x\|$
- $\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in X \\ \|x\| = 1}} \|Tx\|$
  $= \sup_{\substack{x \in X \\ \|x\| = c}} \frac{\|Tx\|}{c} = \sup_{\substack{x \in X \\ \|x\| \leq c}} \frac{\|Tx\|}{c} = \sup_{\substack{x \in X \\ \|x\| < c}} \frac{\|Tx\|}{c}$ $(\forall c > 0)$
- **Equiv boundedness & continuity**: lin. op. $T : X \to Y$:
  T is uniform cts.
  $\iff$ T is cts.
  $\iff$ T is cts. at $\mathbf{0} \in V$
  $\iff$ T is cts. at some $\mathbf{x} \in V$
  $\iff$ T is bounded
- **Cor.**: If lin. op. $T : X \to Y$ is bounded then:
  - $x_n \to x \implies T(x_n) \to T(x)$
  - $\mathrm{Ker}(T)$ is closed (pre-image of closed set on cts. func.)
- **Finite**: Any lin. op. $T : V \to W$ where V finite-dim is cts
- **Thm norm space**: X, Y are normed spaces over the same field:
  - $B(X, Y)$ is a subspace of $L(X, Y)$, and it is a normed space with norm $\|T\|$
  ($B(X, Y)$: bounded lin. ops.; $L(X, Y)$: set of lin. ops)
- **Quotient sp. & projection**: If U closed subspace of V:
  - V/U inherits norm and projection map is cts
  - V Banach $\implies$ V/U Banach
- **Continuous decomposition into projection and isomorphism (tutorial)**: If $T : V \to W$ cts:
  $\overline{T} : V/\mathrm{Ker}(T) \to W$ (s.t. $T = \overline{T} \circ P$, P projection map) injective, cts, $\|\overline{T}\| = \|T\|$
- **Lin. op. is Banach**: If Y is Banach and X is a normed space (not necessarily Banach), then $B(X, Y)$ is Banach
- **(Continuous) dual space** of V: $V^* \coloneqq B(X, \mathbb{R})$
  Algebraic dual space of V: $\mathrm{Hom}(X, \mathbb{R})$ (set of structure-preserving maps from X to $\mathbb{R}$)
- **Thm dual**: The dual space of a normed space is Banach
- **Invertibility**: $T \in B(V, W)$ is invertible $\coloneqq \exists S \in B(V, W)$ such that $TS = \mathrm{Id}_W$ and $ST = \mathrm{Id}_V$
- **Automorphism**:
  $\mathrm{Aut}(V) \coloneqq \{T \in B(V, V) \mid T \text{ is invertible}\}$
  (implies that $T^{-1} \in B(V, V)$ and hence is bounded too)
- $\mathrm{Aut}(V)$ is an open subset of $B(V)$
- **Homeomorphism**: continuous function that has a continuous inverse function
- **Homomorphism**: structure preserving map

Basis

- **Uncountable (algebraic) basis**: If $(V, \|\cdot\|)$ is infinite-dim Banach space, then $\dim V$ (i.e. cardinality of any basis) is uncountable (using the normal (finite/algebraic/Hamel) basis)
- V is **separable**: V has a countable dense subset
- S is a **topological spanning set**: (algebraic) span of S is dense in V
- $(\mathbf{e}_i)_{i \in \mathbb{N}} \in V$ is a **Schauder basis**:
  every $v \in V$ can be expressed as $v = \sum_{i=1}^{\infty} a_i \mathbf{e}_i$, and $\sum_{i=1}^{\infty} a_i \mathbf{e}_i = \sum_{i=1}^{\infty} b_i \mathbf{e}_i \implies \forall i \in \mathbb{N}, a_i = b_i$
- V has a Schauder basis
  $\implies$ V has a countable topological spanning set
  $\implies$ V is separable

Stone-Weierstrass Theorem

- **Continuous functions on compact space is Banach**: If X a compact metric space then $(C(X, \mathbb{R}), \|\cdot\|_{\infty})$ is a Banach space (where $C(X, \mathbb{R})$ is the set of continuous functions from X to $\mathbb{R}$)
- X is an $\mathbb{R}$**-Algebra**: X is a $\mathbb{R}$-vector space and a ring, where the ring addition is equivalent to the vector addition
- $Y \subseteq X$ is a **subalgebra**: vector subspace that is closed under ring multiplication (so Y is also an algebra)
- **Statement**: Let X be a compact metric space.
  If $A \subseteq C(X, \mathbb{R})$ with:
  - $\mathbf{1} \in A$ ($\mathbf{1}$ is the unit of the ring $C(X, \mathbb{R})$)
  - A is a subalgebra
  - A separates points of X
  (i.e. $\forall x_1 \neq x_2 \in X, \exists f \in A$ such that $f(x_1) \neq f(x_2)$)
  Then A is dense in $C(X, \mathbb{R})$ (using $\|\cdot\|_{\infty}$)
  (For $\mathbb{C}$ (instead of $\mathbb{R}$), also require that $f \in A \implies \overline{f} \in A$)
- **Separability of bdd cts functions**: $(C[0,1], \|\cdot\|_{\infty})$ is separable (can be proved with SW by letting A be the set of all polynomials with $\mathbb{R}$ coef., and then observing that the set of all polynomials with $\mathbb{Q}$ coef. is dense in A)

Eigenvalues & Eigenvectors

- **Def**: Given $T : V \to V$ and $v \in V \setminus \{0\}$ and $\lambda \in F$, then: v is an **eigenvector** of T, and λ is an **eigenvalue** of T
- **Finite**: If T is finite, then:
  λ is an eigenvalue $\iff$ $T - \lambda$ is not injective $\iff$
  $T - \lambda$ is not invertible $\iff$ $\det(T - \lambda) = 0$
- **Point spectrum** of T: $\{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of } T\}$
  Spectrum of T: $\sigma(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ not invertible}\}$
  point spectrum of $T \subseteq$ spectrum of T
  Resolvent set of T: $\mathbb{C} \setminus \sigma(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ invertible}\}$
- **Invertibility thms**:
  - Lemma: If $T \in B(V) \coloneqq B(V, V)$ and $\|T\| < 1$ then:
  $I - T$ is invertible, with inverse $I + T + T^2 + \dots \in B(V)$
  - Cor: If $T \in B(V)$ and $\|T\| < |\lambda|$ then $T - \lambda$ is invertible
  (i.e. $\sigma(T)$ is contained in the closed ball of radius $|\lambda|$ centred at origin)
  - Prop: $\mathrm{Aut}(v) \coloneqq \{T \in B(V) \mid T \text{ invertible}\}$ is open in $B(V)$
  - Lemma: If $S, T \in B(V)$ and $ST = TS$ then:
  $S \circ T$ invertible $\implies$ S and T both invertible, with
  $S^{-1} = TR = RT$, $T^{-1} = SR = RS$ where $R = (ST)^{-1}$
  (note: converse only holds when V finite-dim)
- $\sigma(T)$ is a closed subset of $\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\}$ (the closed ball of radius $\|T\|$ centred at origin)
- If $T \in \mathrm{Aut}(V)$, then $\forall \|S\| < \left\|T^{-1}\right\|^{-1}$, $T - S$ is invertible with $\left\|(T - S)^{-1}\right\| \leq \frac{\|T^{-1}\|}{1 - \|T^{-1}\| \cdot \|S\|}$
- **Spectral radius**: $r(T) = \inf_{n \in \mathbb{N}} \|T^n\|^{\frac{1}{n}} \leq \|T\|$
  - Prop: $r(T) = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}}$
  - Prop: $\sigma(T) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq r(T)\}$
- **Spectral mapping thm**: Given any $p(x) \in \mathbb{C}[x]$, then $p(\sigma(T)) = \sigma(p(T))$ (note: we are comparing sets here)

- $\sigma(T) \neq \varnothing$

- λ is an **approximate eigenvalue** of T: $\exists$ unit vectors $(v_n)_{n\in\mathbb{N}} \in V$ such that $(T-\lambda)(v_n) \to 0$ as $n \to \infty$
(0) λ is an eigenvalue $\implies$ λ is an approx. eigenvalue
(1) λ is an approx. eigenvalue $\implies$ $\lambda \in \sigma(T)$
(2) $\lambda \in \partial\sigma(T)$ (boundary of $\sigma(T)$) $\implies$ λ is an approx. eigenvalue

- Note: To find the point spectrum and spectrum of T, find all eigenvalues manually, and find $r(T)$, and reason about the spectrum

Dual Spaces

- **Def**: Given normed space V (not necessarily Banach) over F: Dual space of V: $V^* := B(V, F)$

- V^* is Banach (even though V might not be Banach)

- **Finite**: If $\dim V = n < \infty$ then: $V^* \cong \mathbb{R}^n \cong V$

- **Example**: Given $1 < p < \infty$, then $\ell_p^* \cong \ell_q$ (isometrically isomorphic). Proof sketch:
- Let $T : \ell_q \to \ell_p^*$ where $T(\mathbf{y})(\mathbf{x}) = \sum_i x_i y_i$
- Check that T is well-defined (i.e. the infinite sum converges (from Hölder's))
- Check that $T(\mathbf{y}) : \ell_p \to \mathbb{R}$ is linear
- Check that $T(\mathbf{y})$ is bounded (from Hölder's)
- So $T(\mathbf{y}) \in \ell_p^*$
- Check that T is injective (by showing that $T(\mathbf{y}) = 0 \implies \mathbf{y} = 0$, or show that T is an isometry)
- Check that T is surjective (take any $L \in \ell_p^*$, and let $y_i = L(\mathbf{e}_i)$, then show $\mathbf{y} = (y_i) \in \ell_q$ and $T(\mathbf{y}) = L$)

(3) T is injective: Observe that $\boxed{T(y)(e_i) = y_i}$ (*)
So if $T(y) = 0$, then $y_i = 0 \ \forall i$, ie $y = 0$

(4) T is surjective: this is the main work
Given $L \in l_p^*$, need to find $y \in l_q$ s.t. $T(y) = L$.
Motivated by (*) above, let us set
$y_i = L(e_i)$ (e_i = ith standard basis element)

Claim: $y = (y_i) \in l_q$ & $T(y) = L$.

Strategy for this
Find $\underline{x} \in l_p$ s.t.
$L(\underline{x}) = \sum_i |y_i|^q$
& use
$\|L(x)\| \leq \|L\| \cdot \|x\|_p$

Pf of Claim: For each n, set

$$x_k^{(n)} = \begin{cases} |y_k|^{q-1} \operatorname{sign}(y_k) & \text{if } k \leq n \\ 0 & \text{otherwise} \end{cases}$$

Then $\underline{x}^{(n)} \in l_p$ &

$$\|\underline{x}^{(n)}\|_p = \left[\sum_{k=1}^n |y_k|^{p\cdot(q-1)}\right]^{\frac{1}{p}} = \left[\sum_{k=1}^n |y_k|^q\right]^{\frac{1}{p}} \quad (\text{since } p+q = pq)$$

Moreover, $L(\underline{x}^{(n)}) = \sum_{k=1}^n |y_k|^q$.

So $\sum_{k=1}^n |y_k|^q = L(\underline{x}^{(n)}) \leq \|L\| \cdot \|\underline{x}^{(n)}\|_p = \|L\| \cdot \left[\sum_{k=1}^n |y_k|^q\right]^{\frac{1}{p}}$

$\implies \left(\sum_{k=1}^n |y_k|^q\right)^{\frac{1}{q}} \leq \|L\|$.

$\implies \|y\|_q \leq \|L\|$, ie $y \in l_q$.

Moreover, $L(\underline{x}) = \lim_{n\to\infty} L\left(\sum_{i=1}^n x_i e_i\right) = \sum_{i=1}^\infty x_i L(e_i) = T(y)(\underline{x})$

- (Note: to prove otherwise, use HB to find L such that $L\big|_{\ell_p^*} = 0$ but $L \neq 0$. Then $0 = L\big|_{\ell_p^*} = T(\mathbf{y})$, so $\mathbf{y} = 0$ (by injectivity), so $L = 0$ (contradiction))
- Check that T is an isometry (i.e. $\|T(\mathbf{y})\| = \|\mathbf{y}\|_q \ \forall y \in \ell_q$

- p is **subadditive** if it satisfies:
- $p(v_1 + v_2) \leq p(v_1) + p(v_2)$
- $p(\lambda v) = \lambda p(v) \ \forall \lambda > 0$

- p is **convex** if $\forall 0 \leq t \leq 1, \forall v_1, v_2 \in V$: $p(tv_1 + (1-t)v_2) \leq tp(v_1) + (1-t)p(v_2)$

- subadditive $\implies$ convex

- **Hahn-Banach thm**: Given normed space V (over $\mathbb{R}$ or $\mathbb{C}$) and subspace $W \subseteq V$, for any $F \in W^*$, there exists $\tilde{f} \in V^*$ such that $\tilde{f}\Big|_W = f$ and $\left\|\tilde{f}\right\| = \|f\|$.

- **Corollaries of Hahn-Banach**:
- $v = 0 \iff f(v) = 0 \ \forall f \in V^*$
- $\forall v \in V$, $\exists f_v \in V^*$ such that $\|f_v\| = 1$ and $f_v(v) = \|v\|$ (such an f_v is called a **support functional** of v)
- For $T \in B(V, W)$, the adjoint $T^* \in B(W^*, V^*)$ defined by $T^*(f) := f \circ T$ satisfies: $\|T^*\| = \|T\|$

- **Double dual**: There is a natural map $i : V \to V^{**} := (V^*)^*$ defined by $i(v)(f) := f(v)$. It satisfies:
- $\|i(v)\| \leq \|v\|$
- i is linear
- i is an isometry (and hence continuous and injective)

- V is **reflexive**: i (defined above) is an isomorphism

- reflexive $\implies$ Banach (since V^{**} is Banach)

Baire Category Theorem

- $A \subseteq X$ is **nowhere dense** in X: $\operatorname{Int}(\overline{A}) = \varnothing$ (interior of $\overline{A}$)
$\iff$ $\overline{A}$ contains no non-empty open set
$\iff$ For any non-empty open set $U \subseteq X$, $A \cap U$ is not dense in U
$\iff$ $X \setminus \overline{A}$ is dense in X

- $A \subseteq X$ is **meagre** in X: A is a countable union of nowhere dense subsets of X

- X is of **first category**: X is meagre in X

- X is of **second category**: X is not meagre in X
$\iff$ If $(U_i)_{i\in\mathbb{N}}$ is a countable collection of open dense subsets of X then $\bigcap_i U_{i=1}^\infty \neq \varnothing$
$\iff$ If $X = \bigcup_{n=1}^\infty X_n$ with X_n all closed, then $\exists n \in \mathbb{N}$ such that $\operatorname{Int}(X_n) \neq \varnothing$

- **Baire category theorem**:
X is a complete metric space $\implies$ X is of second category

- **Cor**: X complete and $A \subseteq X$ is of first category
$\implies$ $X \setminus A$ is dense in X

- **Applications**:
- $\mathbb{R} \setminus \mathbb{Q}$ is dense
- Any infinite-dim Banach space has uncountable dimension
- There exists $f \in C[0,1]$ such that f nowhere differentiable (actually, the set of such f is dense in $(C[0,1], \|\cdot\|_\infty)$)
- Given X is a complete metric space, $F \subseteq C(X, \mathbb{R})$ (set of cts functions) such that F is pointwise bounded (i.e. $\forall x \in X$, $\sup_{f\in F} |f(x)| < \infty$), then:
there exists a non-empty open set $U \subseteq X$ such that $\sup_{f\in F, x\in U} |f(x)| < \infty$

Principle of Uniform Boundedness

- **Def**: Given V Banach and W normed and $F \subseteq B(V, W)$:
$\forall v \in V$, $\sup_{T\in F} \|T(v)\| < \infty \implies \sup_{T\in F} \|T\| < \infty$

- **Banach-Steinhaus thm**: Given V Banach and W normed and $(T_n)_{n\in\mathbb{N}}$ a sequence in $B(V, W)$:
$\forall v \in V$, $(T_n(v))_{n\in\mathbb{N}} \to T(v) \in W \implies T \in B(V, W)$

- $T : V \to W$ is **open**: $\forall$ open $U \subseteq V$, $T(U)$ open in W

- **Open mapping theorem**:
Given V, W Banach and $T \in B(V, W)$:
T surjective $\implies$ T open

- **Inverse mapping theorem**:
Given V, W Banach and $T \in B(V, W)$ bijective:
$T^{-1} \in B(W, V)$

- **Closed graph theorem**:
Given V, W Banach and $T : V \to W$ linear map:
T bounded $\iff$ graph of T is closed (w.r.t. $\|(v, w)\| := \max\{\|v\|_V, \|w\|_W\}$)

- **Graph** of T: $\Gamma_T := \{(v, T(v)) \mid v \in V\} \subseteq V \times W$
(when T linear, Γ_T is a vector subspace of $V \times W$)

- Note: Banach-Steinhaus, open mapping, inverse mapping, closed graph have non-linear versions too

Fourier Analysis

- $A := \{\text{finite lin. combin. of } e^{inx} \mid n \in \mathbb{Z}\} \subseteq C(\mathbb{R}/2\pi\mathbb{Z})$
$= \{P(e^{ix}) \mid P \in \mathbb{C}[t]\}$ ($C(\mathbb{R}/2\pi\mathbb{Z})$: cts complex-val fn)

- **Lemma**: A is dense in $C(\mathbb{R}/2\pi\mathbb{Z})$ w.r.t. $\|\cdot\|_\infty$
(and hence $C(\mathbb{R}/2\pi\mathbb{Z})$ separable)

- **Fourier coefficients**: $\hat{f}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$

- $S_N(f)(x) := \sum_{n=-N}^{N} \hat{f}(n)e^{inx}$

- There are some $f \in C(R/2\pi\mathbb{Z})$ such that $S_N(f) \not\to f$ unif.
(so $\{e^{inx} \mid n \in \mathbb{Z}\}$ is not a Schauder basis of $C(\mathbb{R}/2\pi\mathbb{Z})$)

- **Dirichlet's kernel**: $D_N(s) := \sum_{n=-N}^{N} e^{ins}$
Hence $S_N(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_N(x-t)\,dt$
Explicit form: $D_N(s) = \frac{\sin\left(N+\frac{1}{2}\right)s}{\sin\frac{s}{2}}$

- $\frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(s)|\,ds > C\log N$ for some $C > 0$

- $\sigma_N(f) := \frac{1}{N+1}\sum_{n=0}^{N} S_n(f)$

- **Fejér's kernel**: $K_N(s) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(s)$
Hence $\sigma_N(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)K_N(x-s)\,ds$
Explicit form: $K_N(s) = \frac{1}{N+1}\left(\frac{\sin\frac{N+1}{2}s}{\sin\frac{s}{2}}\right)^2$
Properties:
- $K_N(s) \geq 0$
- $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(s)\,ds = 1$
- For any $0 < \delta \ll \pi$, $K_N(s) \xrightarrow{N\to\infty} 0$ unif. for $\delta \leq |s| \leq \pi$
(Note: the last two properties imply that $K_N(s)$ behaves like the Dirac delta function, squished toward 0)

- **Thm**: $\sigma_N(f) \xrightarrow{N\to\infty} f$ unif.
(so $\{e^{inx} \mid n \in \mathbb{Z}\}$ is a topological spanning set of $C(\mathbb{R}/2\pi\mathbb{Z})$)

- **Uniqueness of Fourier series**: Given $f_1, f_2 \in C(\mathbb{R}/2\pi\mathbb{Z})$: $f_1 = f_2 \iff \forall n \in \mathbb{Z}$, $\hat{f}_1(n) = \hat{f}_2(n)$

- $\sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right| < \infty \implies (S_N(f))_{N\in\mathbb{N}}$ is Cauchy in $C(\mathbb{R}/2\pi\mathbb{Z})$ w.r.t $\|\cdot\|_\infty$, and hence $S_N(f) \to f$ unif.

- **Lemma**: $f \in C^k(\mathbb{R}/2\pi\mathbb{Z}) \implies \left|\hat{f}(n)\right| \leq \frac{C}{n^k}$ (k cts diffable)
(hence $f \in C^2(\mathbb{R}/2\pi\mathbb{Z}) \implies S_N(f) \to f$ unif.)

- The completion of $C(\mathbb{R}/2\pi\mathbb{Z})$ is $L^p(\mathbb{R}/2\pi\mathbb{Z}) = L^p[-\pi, \pi]$ so $\hat{f}(n)$ is well-defined for $f \in L^p$.

- $p \geq q \implies L^p(\mathbb{R}/2\pi\mathbb{Z}) \subseteq L^q(\mathbb{R}/2\pi\mathbb{Z})$

- **Riemann-Lebesgue lemma**:
Given $\mathcal{F} : L^1(\mathbb{R}/2\pi\mathbb{Z}) \to \ell_\infty \subseteq \mathbb{C}^{\mathbb{Z}} : f \mapsto \left(\hat{f}(n)\right)_{n\in\mathbb{Z}}$, then $\operatorname{Image}(\mathcal{F}) \subseteq C_0$

- Properties about $\mathcal{F}$:
- $\mathcal{F}$ is injective
- $\mathcal{F}$ is not surjective

- Further results in Fourier analysis:
- $f \in C^1(\mathbb{R}/2\pi\mathbb{Z}) \implies S_N(f) \to f$ unif.
- $\exists f \in L^1(\mathbb{R}/2\pi\mathbb{Z})$ s.t. $(S_N(f)(x))_{N\in\mathbb{N}}$ diverges $\forall x \in [-\pi, \pi]$
- $\exists f \in L^1(\mathbb{R}/2\pi\mathbb{Z})$ s.t. $\|S_N(f) - f\|_1 \not\to 0$ as $N \to \infty$
- If $p > 1$ then $S_N(f) \xrightarrow{ptwise} f$ almost everywhere $f \in L^p$, furthermore $S_N(f) \to f$ w.r.t. $\|\cdot\|_p$

Inner Product Spaces

- **Def**: An **inner product** is a map $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ or $\mathbb{C}$ satisfying:
- $\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$
- $\langle \lambda v, w\rangle = \lambda\langle v, w\rangle$
- $\langle v, \lambda w\rangle = \overline{\lambda}\langle v, w\rangle$
- $\overline{\langle v, w\rangle} = \langle w, v\rangle$
- $\langle v, v\rangle \geq 0$
(it can be inferred that $\langle v, w_1 + w_2\rangle = \langle v, w_1\rangle + \langle v, w_2\rangle$)
Then $(V, \langle\cdot,\cdot\rangle)$ is an **inner product space**

- An inner product gives rise to a norm $\|v\| := \langle v, v\rangle^{\frac{1}{2}}$

- **Cauchy-Schwarz inequality**: $|\langle v, w\rangle| \leq \|v\| \cdot \|w\|$
(or equivalently, $|\langle v, w\rangle|^2 = \langle v, v\rangle \cdot \langle w, w\rangle$)
(or for std. inner prod.: $|\sum_{i=1}^n v_i\overline{w_i}|^2 \leq \sum_{j=1}^n |v_j|^2 \sum_{k=1}^n |w_k|^2$)
with equality iff v and w are lin. dependent (i.e. one of them is zero or one is a scalar multiple of the other)

- **Parallelogram law**:
A norm $\|\cdot\|$ arises from an inner product $\iff$
$\forall v, w \in V$, $2\|v\|^2 + 2\|w\|^2 = \|v + w\|^2 + \|v - w\|^2$

- **Polarisation formula**: If V is an inner product space...
- ... over $\mathbb{R}$: $\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$
$= \frac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right)$
- ... over $\mathbb{C}$: $\langle x, y\rangle = \frac{1}{4}\sum_{\varepsilon=\pm1\pm i} \varepsilon\|x + \varepsilon y\|^2$

- **Cor**:
If $(V, \langle\cdot,\cdot\rangle)$ is an inner product space with arised norm $\|\cdot\|$, and $(\hat{V}, \widehat{\|\cdot\|})$ is the completion of $(V, \|\cdot\|)$,
then $\langle\cdot,\cdot\rangle$ extends to an inner product on $\hat{V}$

- **Hilbert space**: complete inner product space

- **Orthogonality**: x, y are orthogonal $:= \langle x,y\rangle = 0$
orthogonal complement of $X \subseteq V$:
$X^\perp := \{v \in V \mid \langle x,v\rangle = 0\ \forall x \in X\}$
Properties of $X^\perp$:
- $X \subseteq Y \subseteq V \implies X^\perp \supseteq Y^\perp$
- $X^\perp = \overline{\operatorname{Span} X}^\perp$
- $X^\perp$ is a closed subspace of V
- $X \subseteq X^{\perp\perp}$
- $X^\perp = X^{\perp\perp\perp}$
- $X \cap X^\perp = \{0\}$

- **Projection** to $U \subseteq V$: linear map $P : V \to V$ such that $P^2 = P$ and $\operatorname{Image}(P) = U$
(note: implies that $u \in U \implies P(u) = u$)
A projection P of V onto U is equivalent to a subspace $W \subseteq V$ s.t. $U \oplus W = V$. Then $W = \operatorname{Ker}(P)$ and $P : U \oplus W \to U : u + w \mapsto u$.

- **Prop**: If $0 \neq U$ complete subspace of V, then $V = U \oplus U^\perp$

- **Unique closest point lemma**: Given $x \in V$ (inner product space) and a complete subspace $U \subseteq V$, there is a unique $u_0 \in U$ s.t. $d(x,U) := \inf_{u\in U} \|x-u\| = \|x-u_0\|$

- **Orthogonal projection**: Given an inner product space V and a complete subspace $U \subseteq V$, there exists a cts projection $P : V \to V$ s.t. $\operatorname{Image}(P) = U$ and $\operatorname{Ker}(P) = U^\perp$
(furthermore $\|P\| = 1$ and $P(x)$ is the unique point in U closest to x)
Such P is the **orthogonal projection** of V onto U

- **Dual spaces of Hilbert space**:
$\forall v \in V$, let $l_v : V \to \mathbb{C} : u \mapsto \langle u, v\rangle$ (so $l_V \in V^*$)
Let $j : V \to V^* : v \to l_v$
Then j is conjugate linear and isometric
(conjugate linear: $j(v_1 + v_2) = j(v_1) + j(v_2)$ and $j(\lambda v) = \overline{\lambda} j(v)$)
(isometric: $\|j(v)\| = \|v\|$)

- **Riesz representation thm**: j (above) is surjective, and hence j is a conjugate linear isometric isomorphism
(note: it is not actually an isomorphism because of conjugation)

- **Reflexivity of Hilbert space**: V reflexive and V^* Hilbert (with inner product $\langle j(v_1), j(v_2)\rangle_{V^*} = \langle v_2, v_1\rangle_V = \overline{\langle v_1, v_2\rangle_V}$)

Orthonormal Basis

- **Orthogonal system**: $\{e_\alpha\}$ of nonzero elements in $(V, \langle\cdot,\cdot\rangle)$ s.t. $\langle e_\alpha, e_\beta\rangle = 0\ \forall \alpha \neq \beta$

- **Orthonormal system**: orthogonal system with $\|e_\alpha\| = 1\ \forall\alpha$

- Given an orthogonal system S:
S is maximal $\iff \overline{\operatorname{Span}(S)} = V$

- **Orthonormal basis**: maximal orthonormal system

- **Thm**: If V separable Hilbert space then V has a countable orthonormal basis

- **Bessel's inequality**:
If V separable Hilbert space and $\{e_n\}_{n\in\mathbb{N}}$ orthonormal basis then $\forall N \in \mathbb{N}, \forall x \in V, \sum_{i=1}^N |\langle x, e_i\rangle|^2 \leq \|x\|^2$

- **Prop**: If V separable and $\{e_n\}_{n\in\mathbb{N}}$ orthonormal basis then:
- $\forall x \in V$, $x = \sum_{i=1}^\infty \langle x, e_i\rangle e_i$
- $\forall x, y \in V$, $\langle x, y\rangle = \sum_{i=1}^\infty \langle x, e_i\rangle\langle e_i, y\rangle$
- $\forall x \in V$, $\|x\|^2 = \langle x, x\rangle = \sum_{i=1}^\infty |\langle x, e_i\rangle|^2$ (Parseval's identity)

- **Rmk**: Orthogonal basis $\implies$ Schauder basis

- **Riesz-Fischer thm**: Given V separable Hilbert space and $\{e_n\}_{n\in\mathbb{N}}$ orthonormal basis, then:
the linear map $V \to \ell_2 : x \mapsto (\langle x, e_n\rangle)_{n\in\mathbb{N}}$ is a Hilbert space isomorphism (i.e. preserves $\langle\cdot,\cdot\rangle$, and hence must be isometric)
(hence all sep. Hilbert spaces of infinite dim are isomorphic)

- **Fourier analysis results**:
- $\left\{e^{inx}\right\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R}/2\pi\mathbb{Z})$ where $\langle f, g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$
- $\forall f \in L^2$, $S_N(f) \to f$ w.r.t. $\|\cdot\|_2$
- $\forall f \in L^2$, $\|f\|^2 = \sum_{k=-\infty}^{\infty}\left|\hat{f}(k)\right|^2$ (Parseval's identity)
- The map $\mathcal{F} : L^2(\mathbb{R}/2\pi\mathbb{Z}) \to \ell_2 : f \mapsto \left(\hat{f}(n)\right)_{n\in\mathbb{Z}}$ is an isometric isomorphism of Hilbert spaces
- If $f \in C^1(\mathbb{R}/2\pi\mathbb{Z})$ (continuously differentiable functions) then $S_N(f) \to f$ w.r.t. $\|\cdot\|_\infty$

Spectral Theory of Compact Self-Adjoint Operators

- **Isometry**: $T : V \to W$ bounded linear map s.t. $\forall v_1, v_2 \in V$, $\langle T(v_1), T(v_2)\rangle_W = \langle v_1, v_2\rangle_V$

- **Unitary**: $T : V \to W$ bounded linear map that is isometric and surjective
(for finite-dim: $T : V \to V$ unitary $\iff \overline{A}^t A = I$)

- **Bound of T**: $\|T\| = \sup_{\substack{\|x\|=1\\ \|y\|=1}} |\langle T(x), y\rangle|$

- Given $T : V \to W$ a bounded linear map, we have $T^* : W^* \to V^*$ (adjoint map); but due to Riesz representation thm, $V \cong V^*$ and $W \cong W^*$ (up to conjugation only) so we can redefine T^* as $T^* : W \to V$ which is now linear in $\mathbb{C}$ (because the conjugation cancel out)

- $T^* : W \to V$ is characterised by $\langle T(v), w\rangle_W = \langle v, T^*(w)\rangle_V$
(or equivalently $\langle w, T(v)\rangle_W = \langle T^*(w), v\rangle_V$)

- $T^{**} = T : V \to W$ (reminder: T is an inner product space)

- $T : V \to V$ is **self-adjoint**: $T = T^* : V \to V$
Propositions when $T = T^*$:
- $\sigma(T) \subseteq \mathbb{R}$ (in particular, any eigenvalue of T is real)
- Eigenspaces of T for distinct eigenvalues are orthogonal

- **Thm**: If $T = T^*$ then $\|T\| = \sup_{\|x\|=1} |\langle T(x), x\rangle|$

- **Formula**: If $T = T^*$ then:
$\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle = 4\operatorname{Re}\langle T(x), y\rangle$

- **Attainment of approximate eigenvalues**: If $T = T^*$ then either $\|T\|$ or $-\|T\|$ (or both) are approx. eigenvalues (hence $r(T) = \|T\|$)

- **Compact operators**: Given V Banach, $T : V \to V$ is **compact** $:= T(B(1))$ is relatively compact (i.e. $\overline{T(B(1))}$ is compact) (note: $B(1)$ is the closed ball of radius 1)
(equiv. sequential compactness cond.: given any bounded sequence $(x_n) \in V$, $(T(x_n))$ has a convergent subsequence)
Propositions:
- T has finite rank (i.e. $\dim\operatorname{Image}(T) < \infty$) $\implies T$ compact
- The subset of compact operators in $B(V)$ is closed in $B(V)$
- If V is Hilbert, then:
{ compact operators } = closure of { finite-rank operators }

- **Compactness and dual spaces**:
- T compact $\iff T^*$ compact
- T^*T compact $\implies T$ compact

Spectral Theorem

Assume T is self-adjoint and compact in this whole section.

- **Eigenspace** of eigenvalue λ: $N_\lambda := \operatorname{Ker}(T - \lambda I)$

- $\lambda \neq 0$ is an eigenvalue of $T \implies N_\lambda$ is finite-dim

- $\lambda \neq 0$ is an approx. eigenvalue of $T \implies \lambda$ is an eigenvalue of T
(so $\|T\|$ or $-\|T\|$ (or both) must be eigenvalues of T)

- **Distribution of eigenvalues**:
- $\{\lambda \mid \lambda$ is an eigenvalue of $T\}$ is countable
- $\forall\varepsilon > 0$, $\{\lambda \mid |\lambda| > \varepsilon$ and λ is an eigenvalue of $T\}$ is finite
(i.e. the only possible cluster point could be at $\lambda = 0$)

- **Prop**: $V = \overline{\operatorname{Span}\{N_\lambda \mid \lambda \in \sigma_{\mathrm{p}}(T)\}}$ (i.e. span of eigenspaces is dense) (note: $\sigma_{\mathrm{p}}(T) :=$ point spectrum of T)

- **Spectral thm**: Given T compact self-adjoint operator on a Hilbert space V, then:
there exists an orthonormal basis $\{e_n\}$ of V and a sequence of (not necessarily distinct) real numbers $\{a_n\}$ such that:
- $T(e_n) = a_n e_n$
- $\forall\varepsilon > 0$, $|\{a_n \mid |a_n| > \varepsilon\}| < \infty$

- **Cor**: Given T compact self-adjoint operator on a Hilbert space V, then: $\sigma(T) = \overline{\sigma_{\mathrm{p}}(T)} = \sigma_{\mathrm{p}}(T)$ or $\sigma_{\mathrm{p}}(T) \cup \{0\}$
(in particular, $\sigma(T) = \sigma_{\mathrm{p}}(T)$ when $\dim V < \infty$
and $\sigma(T) = \sigma_{\mathrm{p}}(T) \cup \{0\}$ when $\dim V = \infty$)